Comparing Advantages

I was having a conversation recently with a co-worker who was lamenting that the lead on his project didn’t find much value in my friend’s contributions. It seems that the team lead is better on all the various technologies they employ and rarely forgets to remind my friend of that. I pointed out that even if that were true, which I doubt, my friend had an invaluable resource that no one could ever exceed – his time. Every human on this earth has, in a normal day (i.e. one in which they are neither first conceived nor in which they die) the same number of seconds as everyone else. This immutable fact of human existence leads to the idea of what economists call comparative advantage.

What exactly is comparative advantage? It is best explained by examining a simplified version of my friend’s situation. Suppose that we name my friend ‘F’ and his somewhat annoying team lead ‘L’ (I thought about calling him ‘A’ for annoying but I needed it for something else). Also, let’s focus on two tasks that they each can perform and label them ‘A’ and ‘G’.

To understand comparative advantage, we have to say a little about how the each of the persons working on this project employs their skills to perform these tasks. When L says he is better than F he can only mean one of two closely-related things. Either he produces a better result in a fixed amount of time or he produces the same result in a shorter amount of time.

This balance between quality and time-to-complete is present in every daily activity where expertise is involved. Do-it-yourself home projects are excellent examples where the weekend warrior spends two days trying to repair a sink or toilet when the same job could be done in a half-an-hour by a master plumber.

So what doesn’t L just get rid of F? After all, if L’s skill is analogous to the ‘master plumber’ and F’s to the ‘weekend warrior’ why even employ F? This is where the time comes in. Since L has the same amount of time available in a day as F, L can maximize the use of his expertise by relying on F to ‘take work off of his plate’.

Let’s make this more understandable by using concrete numbers. Assume that L can perform task A in 5 hours and with a profit of $50 and task G in 3 hours and results in a profit of $20. Next assume that F can perform task A in 7 hours with a profit of $50 and task G in 4 hours with a profit of $20. Note that in this example, I am imagining a situation where tasks A and G have to be done to a certain level of quality or no profit will result. This is the usual situation when purchasing a good like a computer where the whole thing has to work (would you but a computer that was mostly completed but was missing a hard drive?). There are two final ingredients that have to be considered. First, how many of tasks A and G must the firm of L & F complete? Let’s assume that they need to complete 9 of task A and 8 of task G to live up to demand. Finally assume that L and F are working a 40-hour work week. I will only be examining a single week so I will allow fractions of tasks completed. Let’s summarize the ground rules of our game in a little table.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  | Task A | | Task G | |
| time | profit | time | profit |
| L | 5 | $50 | 3 | $20 |
| F | 7 | $50 | 4 | $20 |

Comparing side-by-side we can see that L is indeed better than F at both tasks. That is to say that L completes both tasks faster than F for the same profit.

If the firm of L&F wants to maximize their profit (and who doesn’t) the operative question is then: what is the proper work load for L and F?

Let’s try a couple of examples. Suppose L & F decide that since L is better at everything they should let him try to maximize the number of tasks completed. It shouldn’t be hard to convince yourself that the maximum number of task L can complete is if he does all nine of the G tasks himself. That leaves just enough time to do 2.6 of the A tasks, bringing L’s output to 11.6 tasks for the week. The resulting work division is shown in the table below netting the firm a profit of$595.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Number of tasks | profit/task | hours/task | total profit | total hours |
| L does A | 2.6 | $50 | 5 | $130 | 13 |
| L does G | 9 | $20 | 3 | $180 | 27 |
|  |  |  |  | $310 | 40 |
| F does A | 5.7 | $50 | 7 | $285 | 39.9 |
| F does G | 0 | $20 | 4 | $0 | 0 |
|  |  |  |  | $285 | 39.9 |

Can their firm actually do better? The answer is yes and it involves having L do fewer tasks, focusing solely on performing A tasks, which have the higher profit margin. The maximum profit the firm can earn is with the following work load

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | Number of tasks | profit/task | hours/task | total profit | total hours |
| L does A | 8 | $50 | 5 | $400 | 40 |
| L does G | 0 | $20 | 3 | $0 | 0 |
|  |  |  |  | $400 | 40 |
| F does A | 0 | $50 | 7 | $0 | 7.7 |
| F does G | 9 | $20 | 4 | $180 | 32 |
|  |  |  |  | $180 | 39.7 |

which results in a net profit of $615 dollars. Observe that in raising the profit of the firm, L actually drops his output from 11.6 tasks to 8 while F drops his profit from $285 to $215. By allowing L to specialize, both L & F stand to make a larger profit than can be made otherwise. And even though F is seemingly inferior to L in all regards, his ability to provide the invaluable resource that is his time allows the firm as a whole to earn more profit. That is comparative advantage.

At this point there are two additional observations to make.

The first is that the example given above is particularly simple in that not only was L better than F at performing both tasks, he has an advantage over F in performing Task A that is greater than his advantage over Fin performing Task G. This advantage is measured as the ratio of L’s profit per hour to F’s profit per hour on a given task. L earns $10/hr on Task A compared to F’s $7.1/hr giving an advantage of 10/7.1 = 1.40. Likewise, L earns $6.7/hr on Task G compared to F’s $5/hr giving an advantage of 6.7/5 = 1.34. The fact that L has the greatest advantage on the task with the greatest profit margin ($10/hr) is what makes the analysis of the maximum profit easy to do.

The situation becomes more nuanced if F can improve in his performance of Task A, completing it in 6 hours instead of 7. In this case, L still has an advantage on Task A at a value of 1.2 but it is now lower his advantage on Task G. With F’s newfound skill, the firm can not only earn more money but they can do so by having L concentrate more on G tasks rather than A. A possible work load is

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | number | profit | hours | total profit | total hours |
| L does A | 3.2 | 50 | 5 | 160 | 16 |
| L does G | 8 | 20 | 3 | 160 | 24 |
|  |  |  |  | 320 | 40 |
| F does A | 6.7 | 50 | 6 | 335 | 40.2 |
| F does G | 0 | 20 | 4 | 0 | 0 |
|  |  |  |  | 335 | 40.2 |

which nets a profit of $655 for the firm. Note that L is still better than F in all regards but L is not only doing more G tasks than before his total number of tasks rises to 11.2. Imagine now a real business with many employees and many tasks and you can see how complex it is to actually determine how to maximize business performance.

The second observation is that there is a human element to the problem of L & F that hasn’t been addressed. I posed a parenthetical question above when I asked who wouldn’t want to maximize their profit. Of course, in that context the profit was defined as dollars. In fact, people define profit in all sorts of different ways - job satisfaction, time off from work, lower stress, etc. – that make maximizing the outcome nearly impossible because of competing agendas and goals . For example, it is possible that the firm of L & F is willing to forego some dollar earnings so that L & F can each share in the tasks being performed. It’s for this reason that when this logic is applied in the broader context where L and F are human institutions (business, governments, or countries) there is a lot of friction in deciding what to do.

A New Direction on Vectors

This post grew out of a discussion I had with my son who is now learning physics at the high school level. He was reading his text book with the old time-honored discussion of vectors being things with directions and magnitudes. This is certainly true for physical vectors that describe things like displacement, velocity, force, torque, and electric and magnetic fields – all the typical menagerie that introductory physics students encounter.

But I couldn’t resist talking to him about vector spaces in general. It has long baffled me why we teach kids from the beginning the dumbed-down version of things then make them unlearn it later on. From what I can tell the really creative thinkers of history were one’s who didn’t succumb to this mind-numbing nonsense. They were people who didn’t let the accepted way of doing things influence them. If they had we would never have heard of them.

Thus I resolved to talk to him about vectors in general. I introduced him to the rules for defining a vector space. A short aside is warranted on this point. Depending on which textbook you read there are subtle differences in what is presented. Some books enumerate only eight properties, some only seven, and the ones that do it best list 10. All of them functionally amount to the same thing but standardization seems to elude the community at large, much to the confusion of students.

I touched on all the abstract definitions and properties just to show that there was a firm structure we could build on, but I emphasized that the prototype example of a vector that one could almost always use and never go wrong was a list. I emphasized that what made the list useful was that it was a vector space in its own right and that almost all other vector spaces could be put into correspondence with a list. I also emphasized that while the list was helpful it is often only a representation of the vector in question and that we shouldn’t be seduced into thinking it was the vector itself. A velocity vector is no more a list of numbers than Bernard Riemann is the list of pixels in the 2-array of light and shadow below.



We finished our discussion by touching on infinite-dimensional vector spaces and I could see that what I really needed was a concrete example that he could play with. After about an hour of back-and-forth I finally came up with something that he seemed to grasp. I offer it here in the hopes that others will be able to use it as well.

What I wanted to create was a model where four properties were met:

* The basic units are lists
* These lists can be added component-wise
* The lists can be infinite
* The lists can be directly translated into a picture show magnitude and direction

A note is needed on the point that the lists can be infinite. By this I mean that the length of the list was limited only by our patience and aesthetics or by the physical memory of the machine. Much in the usual idea of infinity, I wanted a system where the student could always reach into the bag and pull out another dimension (being assured that the student would tire before the computer would).

The model that I created is expressed in the computer algebra system wxMaxima 11.08.0 running Maxima version 5.25.0. I like Maxima for several reasons, chief amongst them being that it is very capable and it is free.

Being based on LISP, Maxima is quite comfortable handling the first property. For the second and third properties, I wanted a way to add lists of different lengths, so that the student would never encounter a limitation saying that one list was as long as another. A different way of expressing that is to say that if there are two lists

they are not actually different in length since we can always imagine that they list only the portions of the list beyond which there are only zeros (i.e. and are the last non-zero entries in their respective lists) so that they actually look like

To do this I wrote a small Maxima function called add\_lists that pads the smallest of the two lists with zeros and then returns the sum. The code for it is

add\_lists(a, b) := block([ret\_lst],

Na : length(a),

Nb : length(b),

delta : abs(Na-Nb),

pad : makelist(0,i,1,delta),

if Na > Nb

then ret\_lst : a + append(b,pad)

else ret\_lst : append(a,pad) + b);

For the final property I decided that a plot of a Fourier series in the interval [0,L] would correspond to the visual addition of the vectors.

The particular choice of Fourier series is the familiar cosine series given by

where the constant offset term given by a\_0 has been set equal to zero and the presence of the term L there to remind us of the interval over which the function is defined. Traditional treatments in advanced mathematics and quantum texts, drive the point that the sine and cosine function are orthogonal polynomials

in a Sturm-Liouville sense and thus are basis vectors in some abstract space. The coeffcients a\_n are then the components of the general vector (function) in this space.

Thus the list is the representation of the vector and the function fourier\_series given by

fourier\_series( lst, L ) := block([expr],

expr : 0,

for i : 1 step 1 thru length(lst) do

( temp\_expr : lst[i]\*cos(2\*%pi\*i\*x/L),

expr : expr + temp\_expr ),

expr);

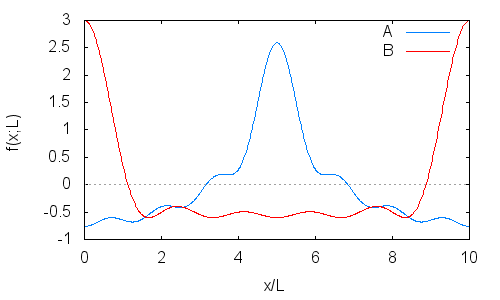
is what delivers the corresponding expression that is plotted.

The final step is to take two lists, plot then each separately to give the ‘direction and magnitude’ of each. Then to add them together plot-wise (that is to say plot the sum of the expressions) and show that the resulting vector is the same that is obtained if the two lists of coefficients are add together first and then plotted.

For a concrete example take the lists

and

Lists A & B produce a plot of and over the interval



With corresponding expansions of fourier\_series(A,L)

and fourier\_series(B,L)

The ‘vector sum’ gives the expression

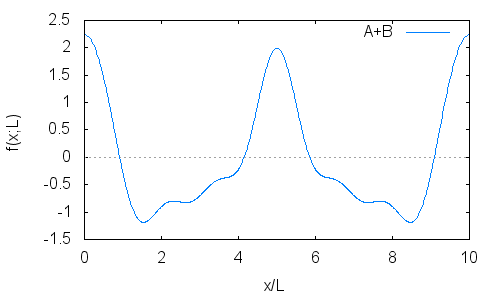
$$-{{\cos \left({{7\,\pi\,x}\over{5}}\right)}\over{7}}+{{\cos \left(

{{6\,\pi\,x}\over{5}}\right)}\over{6}}+{{13\,\cos \left({{4\,\pi\,x

}\over{5}}\right)}\over{20}}+{{4\,\cos \left({{3\,\pi\,x}\over{5}}

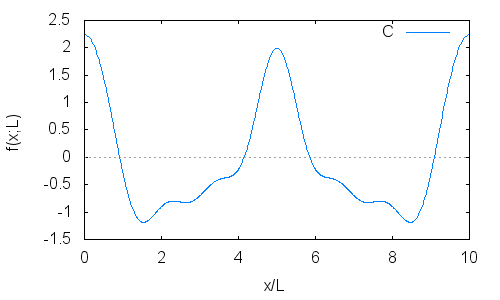
\right)}\over{15}}+{{13\,\cos \left({{2\,\pi\,x}\over{5}}\right)

}\over{10}}$$ and the corresponding plot



The student can then confirm this in terms of adding the list representations together by executing C = add\_list(A,B) which yields

And then evaluating = fourier\_series(C;L) which yields the identical plot



Learning Essentials and Accidentals

There has been a lot of commentary floating around about the recent ‘summoning a demon’ remarks of by Elon Musk on the dangers of artificial intelligence.

<iframe width="560" height="315" src="//www.youtube.com/embed/4BSsgJsmDNs" frameborder="0" allowfullscreen></iframe>

Mr. Musk may be an excellent entrepreneur but his philosophical arts is clearly lacking. Human intelligence has an intrinsic capacity that AI practitioners have yet to capture. This intrinsic capacity is best demonstrated by considering how humans learn their language.

Consider the following images of four fairly common objects: 1) an ordinary magnifying glass, 2) an empty water glass, 3) a water bottle with a magenta filter, and 4) a magenta USB drive.

|  |  |
| --- | --- |
|  |  |
|  |  |

Now imagine that you are about to travel to another country where you don’t speak the native language. You’ve made arrangements to travel on Monday with your interpreter arriving a day later. You pack your bag, making sure to include the four objects shown above (not as strange as it sounds – I routinely travel with three of them).

Due to a strike that happens a few hours after you arrive you become stranded in the country without your interpreter and with no immediate way home. Having nothing better to do you try your hand at learning the language.

You open your bag and pull out the four objects shown above. After a brief pantomime where you open and close your jaw and make sounds in English, one of the native speakers gets the notion that you want to learn some of their mother tongue. Now the fun begins.

The native points at the bottle and says “zerk”. What does this mean? There are many possibilities. He might be saying their word for “container”, or “clear”, or “plastic”, or “water”, or “magenta” or etc. You hope he means “container” and you point at the glass and say “zerk”. He then shakes his head and says “quig”. What does that mean? Maybe it means “empty”, or “glass”, or “clear”, or maybe even “container “.

You open the water bottle and pour all of the water into the glass. Your ad hoc teacher now points at the glass and says “zerk”. Inspired, you then pour some water into your hand, look at him expectantly, and then say “zerk”. Happy that you are now getting the idea, he nods his head vigorously, smiles broadly and says “zerk!” Congratulations you’ve now learned your first word in his language.

Let’s step back for a minute and discuss how you likely put together the concept of water with the sound “zerk”. It was unlikely that for the first attempt at teaching you his language, that your friend would say the word for an accidental associated with the bottle and would focus on the essentials instead. The problem is identifying which properties were essential and which were accidental and then determining which of the hopefully smaller set of essential properties was meant by “zerk”. Somehow you recognized or assumed (implicitly) that an essential property of your interaction with him would be to focus on the essential properties of the objects in question. You also assume that both of you possess a similar way of perceiving the world and culturally categorizing it. Basically, you have an abundance of clues to help you put things together. You also have a capacity to separate out essential properties from accidental ones, so that if the color of the liquid in the bottle were different, or the filter were missing, the bottle would remain a bottle conceptually (even if it looks different).

Now instead of a traveler to an antique land, you are a newborn. Someone is saying “doht” and is pointing at the USB key. What do they mean? How many repetitions and cross-references do you go through in order to figure out what it means? As a newborn, you don’t have the contextual and logical clues that helped you as they did in the foreign travel scenario. But you do have an innate capacity to apprehend the world. Whether it takes a hundred or a thousand attempts, after enough pointing back and forth between the USB key and the bottle, it suddenly hits that “doht” means magenta.

It probably takes longer to realize that a common property shared by the magnifying glass, the drinking glass and the bottle is the property of “clear”. Is this property an essential or accidental property? Most of us would answer it is essential for a magnifying glass to be clear but an accident if the bottle and the drinking glass are clear. So how can any of us learn what clear means when sometimes it falls into one category and sometimes into the other?

I don’t know. I do know that somehow we all can do it. I can’t explain it and I don’t quite know how to describe it any more than I daresay anyone else does, but I know it exists because I witness it.

Now let’s return to the comments of Mr. Musk. It is a sweepingly generous statement to say that our AI efforts have been even primitive successes. As a science, we do not understand this world-apprehending capacity that each of us is equipped with, in firm-ware, as we emerge from the womb. We know we have it but we have yet to codify it let alone translate it into something a computer can emulate. And even if we could, how do we give the machine all the contextual cues and clues that we take for granted in our interactions with our fellow humans. Sometimes I doubt we ever will succeed but for sake of argument I won’t press the point.

What I do know is that we are decades or centuries away from having this capability shared by the silicon and plastic companions that accompany us in our modern life. No Matrix, or Sky-Net, or Demon Seed is just around the corner, ready to burst out of the pentagram to which we attempt to confine it. No man-made machine with sinister or sublime intelligence is ready to sway our world – except for the man-made machines we manufacture through the tried–and-true method of making babies.

One last word is in order. Print out the four pictures featured here in the post, make up some novel sounds for things like “clear”, “small”, “shiny”, and the like, and see how long it takes for a friend to guess. Then give them a turn. It’s actually fun.